A ROTATING STRING SOLUTION AROUND A (3+1)-DIMENSIONAL BLACK HOLE

—Hamilton Formalism—

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(Received December 20, 2004)

Abstract

An attempt is made on studying a classical motion of string around a (3+1)-dimensional static spherical symmetric black hole by the Hamiltonian approach. A gauge fixing is explicitely demonstrated, and a string solution around the (3+1)-dimensional static sphere-symmetric black hole is obtained by the Hamiltonian approach. The string is rotating around the (3+1)-dimensional static black hole.

§ 1. Introduction

In the preceding paper, a rotating string solution around a (2+1)-dimensional static black hole was obtained by the Hamiltonian approach.[1–3]

In this study, a special solution of string around the (3+1)-dimensional static black hole will be obtained only for the case where the string is restricted to a plane by the Hamiltonian approach.

It is well known that the motion of a particle in a gravitational field is dictated by a geodesic equation. It is also known that an observer being infinitely distant from the black hole will see a test particle approaching a horizon but taking an infinitely long time, if the test particle is put outside of the horizon.

Analyzing the motion of a string in the gravitational field is as important as analyzing the motion of a particle in the gravitational field. The motion of the string in the de Sitter universe has been analyzed by H. J. de Vega and N. Sanchez,[4–8] and the motion of the string in a stationary axisymmetric spacetime has been analyzed by V. Frolov *et al.*[9]

In § 2, demonstrating the gauge fixing, the Hamiltonian of string in the (τ, σ), coordinate will be given. In § 3, the Hamiltonian of string in the (x_+, x_-) coordinate will be given by the similar gauge fixing. In § 4, the equation of string motion by the Hamiltonian approach will be given and be solved in a special case in § 5. Finally in § 6, a conclusion will be given.

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§ 2. The Hamiltonian of String in the (τ, σ) Coordinates

Assuming x is the (τ, σ) coordinates, an action S for a string is given by a Polyakov action,

$$S=\int d\tau d\sigma \frac{1}{2\pi\alpha'} \sqrt{-g} g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu . \tag{2.1}$$

Here, $(2\pi\alpha')^{-1}$ represents a string tension, the dynamical variables X^μ represent the background coordinates t, r, θ, φ, and the dynamical variables g_{ab} represent the world sheet metrics $g_{\tau\tau}$, $g_{\tau\sigma}$, $g_{\sigma\sigma}$. $G_{\mu\nu}$ represent the background metric which are given by a Schwarzschild metric as follows

$$G_{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2MG}{r}\right) & & & 0 \\ & \left(1-\frac{2MG}{r}\right)^{-1} & & \\ 0 & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}. \tag{2.2}$$

Here, M is a black hole mass.

From Eqs. (2.1) and (2.2) the Lagrangian density $\mathscr{L}$ and the Lagrangian L are given by

$$\begin{aligned}\mathscr{L}=\frac{1}{2\pi\alpha'\sqrt{-g}}\Big\{&g_{\sigma\sigma}\Big(-\left(1-\frac{2MG}{r}\right)^{-1}\dot{t}^2+\left(1-\frac{2MG}{r}\right)\dot{r}^2+\frac{1}{r^2}\dot{\theta}^2\\&+\frac{1}{r^2\sin^2\theta}\dot{\varphi}^2\Big)+2g_{\sigma\tau}\Big(-\left(1-\frac{2MG}{r}\right)^{-1}\dot{t}(\partial_\sigma t)+\left(1-\frac{2MG}{r}\right)\dot{r}(\partial_\sigma r)\\&+\frac{1}{r^2}\dot{\theta}(\partial_\sigma\theta)+\frac{1}{r^2\sin^2\theta}\dot{\varphi}(\partial_\sigma\varphi)\Big)+g_{\tau\tau}\Big(-\left(1-\frac{2MG}{r}\right)^{-1}(\partial_\sigma t)^2\\&+\left(1-\frac{2MG}{r}\right)(\partial_\sigma r)^2+\frac{1}{r^2}(\partial_\sigma\theta)^2+\frac{1}{r^2\sin^2\theta}(\partial_\sigma\varphi)^2\Big)\Big\},\end{aligned} \tag{2.3}$$

$$L=\int \mathscr{L}\, d\sigma . \tag{2.4}$$

The canonical momenta $p_\mu(\tau, \sigma)$ for the dynamical variables X^μ are defined by

$$p_\mu(\tau, \sigma)=\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu(\tau, \sigma)} . \tag{2.5}$$

Here, dot represents a differentiation with respect to a time τ.

Let P_μ the momenta of a whole string, we generally obtain

$$P_\mu = \int d\sigma p_\mu(\tau, \sigma) . \tag{2.6}$$

From Eqs. (2.3)–(2.5), the canonical momenta for the dynamical variables X^μ are given by

$$p_\mu = -\frac{1}{\sqrt{-g}}(g_{\sigma\sigma}\dot{X}_\mu + g_{\sigma\tau}(\partial_\sigma X_\mu)) . \tag{2.7}$$

First, we note that canonical momenta π_{ab} for the dynamical variables g_{ab} vanish, since the differentiation of g_{ab} with respect to τ is absent in the Lagrangian. The vanishing of canonical momenta can be regarded as a constraint.

Let the constraint ϕ_i be given by

$$\begin{aligned} \phi_1 &= \pi_{\tau\tau} \approx 0 , \\ \phi_2 &= \pi_{\tau\sigma} \approx 0 , \\ \phi_3 &= \pi_{\sigma\sigma} \approx 0 . \end{aligned} \tag{2.8}$$

The Hamiltonian density and the Hamiltonian of string are defined by

$$\mathscr{H} = p_\mu \dot{X}^\mu - \mathscr{L} , \tag{2.9}$$

$$H = \int \mathscr{H} d\sigma . \tag{2.10}$$

From Eqs. (2.7), (2.9), and (2.10), the Hamiltonian of string is given by

$$H = \int d\sigma \left\{ -\frac{\sqrt{-g}}{2g_{\sigma\sigma}}(p_\mu p^\mu + (\partial_\sigma X_\mu)(\partial_\sigma X^\mu)) - \frac{g_{\sigma\tau}}{g_{\sigma\sigma}} p_\mu (\partial_\sigma X^\mu) \right\} . \tag{2.11}$$

From Eq. (2.8), the addition of constraints ϕ_1, ϕ_2, ϕ_3 to the Hamiltonian density $\mathscr{H}$ don't change its value, then the total Hamiltonian density $\mathscr{H}_T$ is defined by

$$\mathscr{H}_T = \mathscr{H} + \sum_{i=1}^{3} u_i \phi_i . \tag{2.12}$$

Here, u_i ($i = 1$–3) are the constants.

The total Hamiltonian H_T is given by

$$H_T = \int d\sigma \left\{ \frac{\sqrt{-g}}{2g_{\sigma\sigma}}(-p_\mu p^\mu - (\partial_\sigma X_\mu)(\partial_\sigma X^\mu)) - \frac{g_{\sigma\tau}}{g_{\sigma\sigma}} p_\mu (\partial_\sigma X^\mu) + \sum_{i=1}^{3} u_i \phi_i \right\} . \tag{2.13}$$

The Poisson bracket $[A, B]$ is defined by

$$[A,\,B]\equiv\sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\right), \tag{2.14}$$

here q_i and p_i represent the canonical variables. Using the Poisson bracket, the Hamilton equations are given by

$$\dot{q}_i=[q_i,\,H]\,, \tag{2.15}$$

$$\dot{p}_i=[p_i,\,H]\,, \tag{2.16}$$

here H represents the Hamiltonian.

Next, we examine the consistency of constrainsts ϕ_i. The constraints ϕ_i are differentiated with respect to the time τ, giving Eqs. (2.14)–(2.16),

$$\begin{aligned}\dot{\phi}_1(\tau,\sigma)&=\left[\phi_1(\tau,\sigma)\,,\int d\sigma'\mathscr{H}_T(\tau,\sigma')\right]\\&=-\frac{\delta H_T}{\delta g_{\tau\tau}(\tau,\sigma)}\\&=-\frac{1}{4\sqrt{-g}}\{p_\mu p^\mu+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu)\}\,,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\dot{\phi}_2(\tau,\sigma)&=\left[\phi_2(\tau,\sigma)\,,\int d\sigma'\mathscr{H}_T(\tau,\sigma')\right]\\&=-\frac{\delta H_T}{\delta g_{\tau\sigma}(\tau,\sigma)}\\&=\frac{g_{\tau\sigma}}{2g_{\sigma\sigma}\sqrt{-g}}\left\{p_\mu p^\mu+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu)+\frac{2\sqrt{-g}}{g_{\tau\sigma}}p_\mu(\partial_\sigma X^\mu)\right\}\,,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\dot{\phi}_3(\tau,\sigma)&=\left[\phi_3(\tau,\sigma)\,,\int d\sigma'\mathscr{H}_T(\tau,\sigma')\right]\\&=-\frac{\delta H_T}{\delta g_{\sigma\sigma}(\tau,\sigma)}\\&=\frac{3g_{\tau\tau}g_{\sigma\sigma}g_{\tau\sigma}^2-2g_{\tau\sigma}^4-g_{\tau\tau}^2g_{\sigma\sigma}^2}{4g_{\sigma\sigma}^2(-g)^{3/2}}\{p_\mu p^\mu+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu)\}\\&\quad-\frac{g_{\sigma\tau}}{g_{\sigma\sigma}^2}p_\mu(\partial_\sigma X^\mu)\,.\end{aligned} \tag{2.19}$$

Since $\phi_i=0$ from Eq. (2.8), the new constraints are derived from Eqs. (2.17) and (2.18). The new ones, ϕ_4 and ϕ_5 are given by

$$\phi_4=p_\mu p^\mu+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu)\approx 0\,, \tag{2.20}$$

$$\phi_5=p_\mu(\partial_\sigma X^\mu)\approx 0\,. \tag{2.21}$$

The constraints ϕ_4 and ϕ_5 are differentiated with respect to the time τ, giving Eqs. (2.22) and (2.23),

$$\dot{\phi}_4(\tau,\sigma)=\left[\phi_4(\tau,\sigma),\int d\sigma'\mathscr{H}_T(\tau,\sigma')\right]$$
$$=\int d\sigma'\left[(g^{\mu\mu})^2\left\{-\frac{2\sqrt{-g}}{g_{\sigma'\sigma'}}(2p_\mu(\sigma)\partial_\sigma X_\mu\partial_\sigma\delta(\sigma-\sigma')+\partial_\sigma(p_\mu(\sigma)\partial_\sigma X_\mu)\delta(\sigma-\sigma'))\right.\right.$$
$$\left.\left.-\frac{2g_{\sigma'\tau}}{g_{\sigma'\sigma'}}\left((p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\partial_\sigma\delta(\sigma-\sigma')+\frac{1}{2}\partial_\sigma(p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\cdot\delta(\sigma-\sigma')\right)\right\}\right], \tag{2.22}$$

$$\dot{\phi}_5(\tau,\sigma)=\left[\phi_5(\tau,\sigma),\int d\sigma'\mathscr{H}_T(\tau,\sigma')\right]$$
$$=\int d\sigma'\left[(g^{\mu\mu})^2\left(-\frac{\sqrt{-g}}{g_{\sigma'\sigma'}}\right)\left\{(p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\partial_\sigma\delta(\sigma-\sigma')\right.\right.$$
$$\left.+\frac{1}{2}\partial_\sigma(p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\delta(\sigma-\sigma')\right\}$$
$$\left.+(g^{\mu\mu})^2\left(-\frac{g_{\sigma'\tau}}{g_{\sigma'\sigma'}}\right)\{2p_\mu(\sigma)\partial_\sigma X_\mu\partial_\sigma\delta(\sigma-\sigma')+\partial_\sigma(p_\mu(\sigma)\partial_\sigma X_\mu)\delta(\sigma-\sigma')\}\right]. \tag{2.23}$$

The right hand sides of Eqs. (2.22) and (2.23) are zero from Eqs. (2.20) and (2.21), and no new condition is obtained. The 2nd and 3rd differentiations of ϕ_i with respect to the time τ give no new condition.

The calculations of Poisson bracket $[\phi_i(\sigma),\phi_j(\sigma')]$ easily show that all except for $[\phi_4,\phi_5]$ vanish.

Poisson bracket $[\phi_4(\sigma),\phi_5(\sigma')]$ is calculated to be

$$[\phi_4(\sigma),\phi_5(\sigma')]$$
$$=[p_\mu(\sigma)p^\mu(\sigma)+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu),\ p_\mu(\sigma')\partial_{\sigma'}X^\mu]$$
$$=2(g^{\mu\mu})^2\left\{(p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\partial_\sigma\delta(\sigma-\sigma')+\frac{1}{2}\partial_\sigma(p_\mu(\sigma)^2+\partial_\sigma X_\mu^2)\delta(\sigma-\sigma')\right\}. \tag{2.24}$$

The right hand side of Eq. (2.24) is zero from Eqs. (2.20) and (2.21).

Accordingly,

$$[\phi_i,\phi_j]=0\,, \tag{2.25}$$

$$[\phi_i,H_T]=0\,, \tag{2.26}$$

are fulfilled, and thus the constraint ϕ_i are the primary constraints.

Then, let the secondary constraints x_i be given by

$$\chi_1=g_{\tau\tau}+1\approx 0\,, \tag{2.27}$$

$$\chi_2=g_{\tau\sigma}\approx 0\,, \tag{2.28}$$

$$\chi_3=g_{\sigma\sigma}-1\approx 0\,, \tag{2.29}$$

$$\chi_4=ct(\tau,\sigma)-MG\tau\approx 0\,, \tag{2.30}$$

$$\chi_5=f(r,\theta,\sigma)\approx 0\,. \tag{2.31}$$

Eqs. (2.27)–(2.29) are constraints derived from the metric of string. Eq. (2.30) is necessary for equalizing t and τ. The number of secondary constraints can be selected as same as those of primary constraints. Then, the 5th secondary constraint is given by Eq. (2.31).

A determinant $D_{gg'}$ is defined by

$$D_{gg'}=[\chi_g(\tau,\sigma)\,,\ \phi_{g'}(\tau,\sigma'')]\,, \tag{2.32}$$

which represents Poisson bracket for the secondary constraints x_g and the primary constraints $\phi_{g'}$.

Calculation of Eq. (2.32) gives

$$D_{gg'}=\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \dfrac{-2cp_0}{\left(1-\dfrac{2MG}{r}\right)} & c(\partial_\sigma t) \\ & 0 & & 2p_1\dfrac{\partial f}{\partial r}\left(1-\dfrac{2MG}{r}\right) & \dfrac{\partial f}{\partial r}(\partial_\sigma r)+\dfrac{\partial f}{\partial \theta}(\partial_\sigma \theta) \end{pmatrix}\delta(\sigma-\sigma')\,. \tag{2.33}$$

Accordingly,

$$\det D\neq 0\,, \tag{2.34}$$

which shows that the secondary constraints x_i ($i=1$–5) can be fixed as the gauge conditions.

Eqs. (2.20) and (2.21) ensure that the total Hamiltonian density $\mathscr{H}_T$ is unchanged upon the addition of constraints ϕ_4 and ϕ_5. Then, a generalized Hamiltonian density $\mathscr{H}_G$ is defined by

$$\mathscr{H}_G=\mathscr{H}+\sum_{i=1}^{5} u_i\phi_i\,. \tag{2.35}$$

Here, u_i ($i=1$–5) are the constants.

A generalized Hamiltonian H_G is given by

$$H_G=\int d\sigma\Big\{-\frac{\sqrt{-g}}{2g_{\sigma\sigma}}(p_\mu p^\mu+(\partial_\sigma X_\mu)(\partial_\sigma X^\mu))-\frac{g_{\sigma\tau}}{g_{\sigma\sigma}}p_\mu(\partial_\sigma X^\mu)+\sum_{i=1}^{5}u_i\phi_i\Big\}. \tag{2.36}$$

To determine the constants u_i, we differentiate the secondary constraints x_i with respect to the time τ.

$$\begin{aligned}\dot{\chi}_4(\tau,\sigma)&=[t(\tau,\sigma)-MG\tau\,,\ H_G]+\frac{\partial\chi_4}{\partial\tau}\\&=\Big\{-\Big(-\frac{\sqrt{-g}}{2g_{\sigma\sigma}}+u_4\Big)\Big(1-\frac{2MG}{r}\Big)^{-1}2p_0+\Big(-\frac{g_{\sigma\tau}}{g_{\sigma\sigma}}+u_5\Big)\partial_\sigma t\Big\}-MG\\&\approx 0\,,\end{aligned} \tag{2.37}$$

$$\begin{aligned}\dot{\chi}_5(\tau,\sigma)&=[\,f(r,\theta,\sigma)\,,\ H_G]\\&=\Big(-\frac{\sqrt{-g}}{2g_{\sigma\sigma}}+u_4\Big)\Big\{\frac{\partial f}{\partial r}\Big(1-\frac{2MG}{r}\Big)2p_1+\frac{\partial f}{\partial\theta}\frac{2p_2}{r^2}\Big\}\\&\quad+\Big(-\frac{g_{\sigma\tau}}{g_{\sigma\sigma}}+u_5\Big)\Big(\frac{\partial f}{\partial r}\partial_\sigma r+\frac{\partial f}{\partial\theta}\partial_\sigma\theta\Big)\approx 0\,.\end{aligned} \tag{2.38}$$

Using Eqs. (2.37) and (2.38), the constants u_4 and u_5 are given and the generalized Hamiltonian H_G is determined.

§ 3. The Hamiltonian of String in the (x_+, x_-) Coordinates

In this section, the Hamiltonian of string in the (x_+, x_-) coordinates defined by

$$x_\pm=\frac{1}{\sqrt{2}}(\tau\pm\sigma)\,, \tag{3.1}$$

will be obtained as in the case of the (τ,σ) coordinates described in the preceding section.

Assuming x is the (x_+, x_-) coordinates, the action S for the string is

$$S=\int dx_+dx_-\frac{1}{2}\sqrt{-g}g^{ab}G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\,. \tag{3.2}$$

Here, the dynamical variables X^μ represent the background coordinates t, r, θ, φ, and g_{ab} represent the string world sheet metrics g_{++}, g_{+-}, g_{--}. $G_{\mu\nu}$ represent the background metric, and in a (3+1)-dimensional static black hole metric they are given by Eq. (2.2).

From Eqs. (3.2) and (2.2), the Lagrangian density $\mathscr{L}$ and the Lagrangian L are

$$\mathscr{L}=\frac{-G_{\mu\mu}}{2\sqrt{-g}}(g_{--}(\partial_+X^\mu)^2+2g_{+-}\partial_+X^\mu\partial_-X^\mu+g_{++}(\partial_-X^\mu)^2)\,, \tag{3.3}$$

$$L=\int\mathscr{L}\,d\sigma\,. \tag{3.4}$$

The canonical momenta $p_\mu(x_+, x_-)$ for the dynamical variables X^μ are defined by

$$p_\mu(x_+, x_-)=\frac{\delta\mathscr{L}}{\delta(\partial_+X^\mu(x_+, x_-))}\,. \tag{3.5}$$

Here, ∂_+ represents the differentiation with respect to the time x_+.

Let p_μ the momenta of the whole string, we generally obtain

$$P_\mu=\int dx_-p_\mu(x_+, x_-)\,. \tag{3.6}$$

From Eqs. (3.3)–(3.5), the canonical momenta for the dynamical variables X^μ are given by

$$p_\mu=\frac{-G_{\mu\mu}}{\sqrt{-g}}(g_{--}\partial_+X^\mu+g_{+-}\partial_-X^\mu)\,. \tag{3.7}$$

First, we note that canonical momenta π_{ab} for the dynamical variables g_{ab} vanish, since the differentiation of g_{ab} with respect to τ are absent in the Lagrangian. The vanishing of canonical momenta can be regarded as constraints.

Let those constraints ϕ_i be given by

$$\begin{aligned}\phi_1&=\pi_{++}\approx 0\,,\\ \phi_2&=\pi_{+-}\approx 0\,,\\ \phi_3&=\pi_{--}\approx 0\,.\end{aligned} \tag{3.8}$$

The Hamiltonian density and the Hamiltonian of string are

$$\mathscr{H}=p_\mu(\partial_+X^\mu)-\mathscr{L}\,, \tag{3.9}$$

$$H=\int\mathscr{H}\,dx_-\,. \tag{3.10}$$

From Eqs. (3.3), (3.9), and (3.10), the Hamiltonian of the string is calculated as

$$H=\int dx_-\left\{-\frac{\sqrt{-g}}{2g_{--}}(p_\mu p^\mu+(\partial_-X_\mu)(\partial_-X^\mu))-\frac{g_{+-}}{g_{--}}p_\mu(\partial X^\mu)\right\}. \tag{3.11}$$

From Eq. (3.8), the addition of constraints ϕ_1, ϕ_2, ϕ_3 to the Hamiltonian density $\mathscr{H}$ don't change its value, then the total Hamiltonian density $\mathscr{H}_T$ is defined by

$$\mathscr{H}_T=\mathscr{H}+\sum_{i=1}^{3} u_i\phi_i\,. \tag{3.12}$$

Here, u_i ($i=1$–3) are the constants.

The total Hamiltonian $\mathscr{H}_T$ is

$$H_T=\int dx_-\left\{\frac{\sqrt{-g}}{2g_{--}}(-p_\mu p^\mu-(\partial_-X_\mu)(\partial_-X^\mu))-\frac{g_{+-}}{g_{--}}p_\mu(\partial_-X^\mu)+\sum_{i=1}^{3}u_i\phi_i\right\}. \tag{3.13}$$

Next, we examine the consistency of constraints ϕ_i ($i=1$–3). The constraints ϕ_i are differentiated with respect to the time x_+, giving Eqs. (3.14)–(3.16).

$$\begin{aligned}\partial_+\phi_1&=[\phi_1(x_-),H_T(x'_-)]\\&=-\frac{\partial\pi_{++}}{\partial\pi_{++}}\frac{\partial H_T}{\partial g_{++}}\\&=-\int dx_-\frac{1}{4\sqrt{-g}}(p_\mu p^\mu+\partial_-X_\mu\partial_-X^\mu)\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\partial_+\phi_2&=[\phi_2(x_-),H_T(x'_-)]\\&=-\frac{\partial\pi_{+-}}{\partial\pi_{+-}}\frac{\partial H_T}{\partial g_{+-}}\\&=\int dx_-\frac{g_{+-}}{2\sqrt{-g}g_{--}}(p_\mu p^\mu+\partial_-X_\mu\partial_-X^\mu)+\frac{1}{g_{--}}p_\mu\partial_-X^\mu\,,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\partial_+\phi_3&=[\phi_3(x_-),H_T(x'_-)]\\&=-\frac{\partial\pi_{--}}{\partial\pi_{--}}\frac{\partial H_T}{\partial g_{--}}\\&=-\int dx_-\left\{\left(\frac{g_{++}}{4\sqrt{-g}g_{--}}+\frac{\sqrt{-g}}{2g_{--}^2}\right)(p_\mu p^\mu+\partial_-X_\mu\partial_-X^\mu)+\frac{g_{+-}}{g_{--}^2}p_\mu\partial_-X^\mu\right\}.\end{aligned} \tag{3.16}$$

Since $\phi_i=0$ ($i=1$–3) from Eq. (3.8), the new constraints are derived from Eqs. (3.14) and (3.15). The new ones, ϕ_4 and ϕ_5 are

$$\phi_4=p_\mu p^\mu+(\partial_-X_\mu)(\partial_-X^\mu)\approx 0\,, \tag{3.17}$$

$$\phi_5=p_\mu(\partial_-X^\mu)\approx 0\,. \tag{3.18}$$

Constraints ϕ_4 and ϕ_5 are differentiated with respect to the time x_+, giving Eqs. (3.19) and (3.20),

$$\begin{aligned}
\partial_+\phi_4 &= [\phi_4(x_-), H_T(x'_-)] \\
&= \int dx'_- \Big[-\frac{\sqrt{-g(x'_-)}}{g_{--}(x'_-)} \{4p_\mu(x_-)\partial_- X^\mu(x_-)\partial_-\delta(x_- - x'_-) \\
&\quad + 2\partial_-(p_\mu\partial_- X^\mu)\cdot\delta(x_- - x'_-)\} \\
&\quad - \frac{g_{+-}(x'_-)}{g_{--}(x'_-)} \{2\partial_-\delta(x_- - x'_-)(p_\mu p^\mu + \partial_- X_\mu \partial_- X^\mu) \\
&\quad + \delta(x_- - x'_-)\cdot\partial_-(p_\mu p^\mu + \partial_- X_\mu\partial_- X^\mu)\} \Big],
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\partial_+\phi_5 &= [\phi_5(x_-), H_T(x'_-)] \\
&= \int dx'_- \Big[\Big(-\frac{\sqrt{-g(x'_-)}}{g_{--}(x'_-)} \Big) \Big\{ \partial_-\delta(x_- - x'_-)(p_\mu p^\mu + \partial_- X_\mu \partial_- X^\mu) \\
&\quad + \frac{1}{2}\partial_-(\partial_- X_\mu \partial_- X^\mu + p_\mu p^\mu)\delta(x_- - x'_-) \Big\} \\
&\quad + \Big(-\frac{g_{+-}(x'_-)}{g_{--}(x'_-)} \Big) \{2\partial_- X^\mu p_\mu \partial_-\delta(x_- - x'_-) + \partial_-(\partial_- X^\mu p_\mu)\delta(x_- - x'_-)\} \Big].
\end{aligned} \tag{3.20}$$

The right hand sides of Eqs. (3.19) and (3.20) are zero from Eqs. (3.17) and (3.18), and no new condition is obtained. The 2nd and 3rd differentiations of ϕ_i with respect to the time x_+ give no new condition.

The calculations of Poisson bracket $[\phi_i(x_-), \phi_j(x'_-)]$ easily show that all except for $[\phi_4, \phi_5]$ vanish.

The Poisson bracket $[\phi_4(x_-), \phi_5(x'_-)]$ is calculated to be

$$\begin{aligned}
&[\phi_4(x_-), \phi_5(x'_-)] \\
&= [G^{\mu\mu}p_\mu(x_-)^2,\ p_\mu(x'_-)\partial_-{}'X^\mu(x'_-)] + [G_{\mu\mu}(\partial_- X^\mu(x_-))^2,\ p_\mu(x'_-)\partial_-{}'X^\mu(x'_-)] \\
&= 2\partial_-\delta(x_- - x'_-)(p_\mu p^\mu + \partial_- X_\mu \partial_- X^\mu) + \partial_-(p_\mu p^\mu + \partial_- X_\mu \partial_- X^\mu)\delta(x_- - x'_-).
\end{aligned} \tag{3.21}$$

The right hand side of Eq. (3.21) is zero from Eqs. (3.17) and (3.18).

Accordingly,

$$[\phi_i, \phi_j] = 0, \tag{3.22}$$

$$[\phi_i, H_T] = 0, \tag{3.23}$$

are fulfilled, and thus the constraints ϕ_i ($i=1$–5) are the primary constraints.

Then, let the secondary constraints x_i be given by

$$\chi_1 = g_{++} \approx 0, \tag{3.24}$$

$$\chi_2 = g_{+-} + 1 \approx 0\,, \tag{3.25}$$

$$\chi_3 = g_{--} \approx 0\,, \tag{3.26}$$

$$\chi_4 = ct(x_+, x_-) - \frac{MG}{\sqrt{2}}(x_+ + x_-) \approx 0\,, \tag{3.27}$$

$$\chi_5 = \theta - x_- \approx 0\,. \tag{3.28}$$

Eqs. (3.24)–(3.28) are the constraints derived from the metric of string. Eq. (3.27) is necessary for equalizing t and x_-. The number of secondary constraints is same as those of primary constraints. Then, the 5th secondary constraint is given by Eq. (3.28).

The determinant $D_{gg'}$ is defined by

$$D_{gg'} = [\chi_g(x_+, x_-)\,,\ \phi_{g'}(x_+, x_-'')]\,, \tag{3.29}$$

which represents the Poisson bracket for the secondary constraints x_g and the primary constraints $\phi_{g'}$.

The calculation of Eq. (3.29) gives

$$D_{gg'} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & \dfrac{-2cp_0}{\left(1-\dfrac{2MG}{r}\right)} & c(\partial_- t) \\ & 0 & \dfrac{2p_2}{r^2} & \partial_- \theta \end{pmatrix} \delta(x_- - x_-')\,. \tag{3.30}$$

Accordingly,

$$\det D \neq 0\,, \tag{3.31}$$

which shows that the secondary constraints x_i ($i=1$–5) can be fixed as the gauge conditions.

Eqs. (3.17) and (3.18) ensure that the total Hamiltonian density $\mathscr{H}_T$ is unchanged upon the addition of constraints ϕ_4 and ϕ_5. Then, the generalized Hamiltonian density $\mathscr{H}_G$ is defined by

$$\mathscr{H}_G = \mathscr{H} + \sum_{i=1}^{5} u_i \phi_i\,. \tag{3.32}$$

Here, u_i ($i=1$–5) are the constants.

The generalized Hamiltonian H_G is

$$H_G=\int dx_-\left\{-\frac{\sqrt{-g}}{2g_{--}}(p_\mu p^\mu+(\partial_-X_\mu)(\partial_-X^\mu))-\frac{g_{+-}}{g_{--}}p_\mu(\partial X^\mu)+\sum_{i=1}^{5}u_i\phi_i\right\}. \tag{3.33}$$

To determine the constants u_i, we differentiate the secondary constraints χ_i ($i=1$–5) with respect to the time τ.

$$\dot{\chi}_4(x_+,x_-)=\left[\left(t(x_+,x_-)-\frac{MG}{\sqrt{2}}(x_++x_-)\right),\ H_G\right]+\frac{\partial\chi_4}{\partial x_+}$$

$$=c\left\{-\frac{2p_0\left(-\frac{\sqrt{-g}}{2g_{--}}+u_4\right)}{\left(1-\frac{2MG}{r}\right)}+\left(-\frac{g_{+-}}{g_{--}}+u_5\right)(\partial_-t)\right\}-\frac{MG}{\sqrt{2}}\approx 0\,, \tag{3.34}$$

$$\dot{\chi}_5(x_+,x_-)=[(\theta-x_-),\ H_G]$$

$$=\frac{2p_2}{r^2}\left(-\frac{\sqrt{-g}}{2g_{--}}+u_4\right)+\left(-\frac{g_{+-}}{g_{--}}+u_5\right)(\partial_-\theta)\approx 0\,. \tag{3.35}$$

Using Eqs. (3.34) and (3.35), the constants u_4 and u_5 are determined by

$$u_4=\frac{\sqrt{-g}}{2g_{--}}-\frac{1}{2}\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2MG}{r}\right)MG}\right)^{-1}, \tag{3.36}$$

$$u_5=\frac{p_2}{r^2}\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2MG}{r}\right)MG}\right)^{-1}+\frac{g_{+-}}{g_{--}}. \tag{3.37}$$

From $\dot{\chi}_i$ $(i=1$–$3)=0$,

$$u_i=0\quad (i=1\text{–}3)\,. \tag{3.38}$$

Since u_i ($i=1$–5) have been determined from Eqs. (3.36)–(3.38), the generalized Hamiltonian H_G is given by

$$H_G=-\left(\frac{p_2}{r^2}+\frac{\sqrt{2}p_0}{\left(1-\frac{2MG}{r}\right)MG}\right)^{-1}$$

$$\times\left(\frac{1}{2}p_\mu p^\mu+\frac{1}{2}\partial_-X_\mu\partial_-X^\mu-\frac{p_2}{r^2}p_\mu\partial_-X^\mu\right). \tag{3.39}$$

§ 4. The Equation of String Motion by Hamiltonian Approach

The Hamilton equations of string are

$$\dot{t}=\left[t(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.1}$$

$$\dot{r}=\left[r(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.2}$$

$$\dot{\theta}=\left[\theta(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.3}$$

$$\dot{\varphi}=\left[\varphi(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.4}$$

$$\dot{p}_0=\left[p_0(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.5}$$

$$\dot{p}_1=\left[p_1(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.6}$$

$$\dot{p}_2=\left[p_2(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right], \tag{4.7}$$

$$\dot{p}_3=\left[p_2(x_+, x_-), \int dx'_- H_G(x_+, x'_-)\right]. \tag{4.8}$$

The calculation of Eqs. (4.1)–(4.8) gives

$$\frac{1}{2}p_\mu p^\mu+\frac{1}{2}\partial_- X_\mu \partial_- X^\mu-\frac{p_2}{r^2}p_\mu \partial_- X^\mu=0\,, \tag{4.9}$$

$$\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)\frac{\partial r}{\partial x_+}=-\left(1-\frac{2GM}{r}\right)p_1+\frac{p_2}{r^2}\frac{\partial r}{\partial x_-}\,, \tag{4.10}$$

$$\frac{1}{2}p_\mu p^\mu+\frac{1}{2}\partial_- X_\mu \partial_- X^\mu+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}p_\mu \partial_- X^\mu=0\,, \tag{4.11}$$

$$\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)\frac{\partial \varphi}{\partial x_+}=-\frac{p_3}{r^2\sin^2\theta}+\frac{p_2}{r^2}\frac{\partial \varphi}{\partial x_-}\,, \tag{4.12}$$

$$\frac{\partial p_0}{\partial x_+}=\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(\left(1-\frac{2GM}{r}\right)\frac{\partial t}{\partial x_-}+\frac{p_2 p_0}{r^2}\right)\right\}, \tag{4.13}$$

$$\frac{\partial p_1}{\partial x_+}=\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(-\left(1-\frac{2GM}{r}\right)^{-1}\frac{\partial r}{\partial x_-}+\frac{p_1 p_2}{r^2}\right)\right\}, \tag{4.14}$$

$$\frac{\partial p_2}{\partial x_+}=-\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(-r^2\frac{\partial \varphi}{\partial x_-}+\frac{p_2{}^2}{r^2}\right)\right\}. \tag{4.15}$$

$$\frac{\partial p_3}{\partial x_+}=\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(-r^2\sin^2\varphi\frac{\partial\theta}{\partial x_-}+\frac{p_2p_3}{r^2}\right)\right\}. \tag{4.16}$$

Eqs. (4.9) and (4.16) give the constraints of the string as follows,

$$p_\mu\partial_-X^\mu=0\,, \tag{4.17}$$

$$p_\mu p^\mu+\partial_-X_\mu\partial_-X^\mu=0\,. \tag{4.18}$$

In a previous paper,[1] the secondary constraints x_i ($i=4, 5$) was selected as

$$x_4=t-\frac{MG}{\sqrt{2}}(x_++x_-)\approx 0\,, \tag{4.19}$$

$$x_5=\theta-x_-\approx 0\,. \tag{4.20}$$

As Eqs. (4.22) and (4.23) give

$$\frac{\partial t}{\partial x_-}=\frac{MG}{\sqrt{2}}\,, \tag{4.21}$$

$$\frac{\partial\theta}{\partial x_-}=1\,, \tag{4.22}$$

the equations of motion of string are derived as follows,

$$\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)\frac{\partial r}{\partial x_+}=-\left(1-\frac{2GM}{r}\right)p_1+\frac{p_2}{r^2}\frac{\partial r}{\partial x_-}\,, \tag{4.23}$$

$$\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)\frac{\partial\varphi}{\partial x_+}=-\frac{p_3}{r^2\sin^2\theta}+\frac{p_2}{r^2}\frac{\partial\varphi}{\partial x_-}\,, \tag{4.24}$$

$$\frac{\partial p_0}{\partial x_+}=\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(\left(1-\frac{2GM}{r}\right)\frac{MG}{\sqrt{2}}+\frac{p_2p_0}{r^2}\right)\right\}, \tag{4.25}$$

$$\frac{\partial p_1}{\partial x_+}=\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2}+\frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(-\left(1-\frac{2GM}{r}\right)^{-1}\frac{\partial r}{\partial x_-}+\frac{p_1p_2}{r^2}\right)\right\}, \tag{4.26}$$

$$\frac{\partial p_2}{\partial x_+} = -\frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2} + \frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(r^2 + \frac{p_2{}^2}{r^2}\right)\right\}, \tag{4.27}$$

$$\frac{\partial p_3}{\partial x_+} = \frac{\partial}{\partial x_-}\left\{\left(\frac{p_2}{r^2} + \frac{\sqrt{2}\,p_0}{\left(1-\frac{2GM}{r}\right)MG}\right)^{-1}\left(-r^2\sin^2\varphi\,\frac{\partial\theta}{\partial x_-} + \frac{p_2 p_3}{r^2}\right)\right\}, \tag{4.28}$$

$$x_4 = t - \frac{MG}{\sqrt{2}}(x_+ + x_-) \approx 0\,, \tag{4.29}$$

$$x_5 = \theta - x_- \approx 0\,. \tag{4.30}$$

§ 5. The Special Solution of String Motion

The equations and constraints of string motion obtained in § 4 can be solved in a special case.

Being the string restricted to a plane as follows,

$$\varphi = 0\,, \tag{5.1}$$

which derives the canonical momentum p_2 as follows,

$$p_3 = 0\,, \tag{5.2}$$

from Eq. (4.16). Eqs. (5.1) and (5.2) fulfill Eq. (4.16).

Being the background coordinate r of the string given by the constant, the canonical momentum p_1 is derived from Eq. (4.10) as follows,

$$p_1 = 0\,, \tag{5.3}$$

which fulfills Eq. (4.14). Being p_0 and p_2 given by a constant, which fulfills Eq. (4.23) and (4.13).

Let the canonical momenta p_2 be given by

$$p_2 = kr^2\,, \tag{5.4}$$

here k is the constant. From Eqs. (5.3), (5.4), and (4.17), the canonical momenta p_0 is derived as follows,

$$p_0 = -\frac{\sqrt{2}\,kr^2}{MG}\,. \tag{5.5}$$

From the calculations of Eqs. (5.3)–(5.5) and (4.18), the equation of string is derived as follows,

$$r^6 - \frac{M^2G^2}{2}\left(\frac{1}{k^2}+1\right)r^4 + M^3G^3\left(\frac{1}{k^2}+1\right)r^3 + \frac{M^4G^4}{4k^2}(r-2MG)^2 = 0\,. \tag{5.6}$$

From the numerical calculations, the solution for this equation exists when k is sufficiently small, e.g. $k=10^{-2}$. No solution exists when k is infinite. This situation holds for the (3+1)-dimensional static black hole as well as the (2+1)-dimensional one.

In this study, the solution of the string around the (3+1)-dimensional static black hole is easily derived using that around the (2+1)-dimensional one.

The obtained string solution is $r=\text{const.}$, $\varphi=0$, $p_1=0$, $p_3=0$, $p_2=kr^2$ (k: constant) and $p_0=-\dfrac{\sqrt{2}}{MG}p_2$, which is the slowly rotating string around the (3+1)-dimensional static black hole as well as the (2+1)-dimensional one.

Further study will be concentrated on the another special solution of the string around the (3+1)-dimensional static black hole.

Acknowledgements

The author would like to thank Prof. A. Hosoya (Tokyo Institute of Technology) for many useful discussions and helpful suggestions.

The author would like to express her sincere gratitude to Prof. K. Kawarabayashi (Tokyo International University) and Prof. K. Yazaki (Tokyo Woman's Christian University) for encouragements throughout the course of this research.

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